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MECHANICAL FOUNDATIONS OF BIREFRINGENCE OF PHOTO-ELASTOPLASTIC MEDIA

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Abstract—General birefringent formulae for an optically elastoplastic medium are theoretically deduced from the mechanical point of view. A hypothetical photo-elastoplastic medium is defined, i.e. the index tensor is a function of the elastic strain tensor and the plastic strain tensor. According to the principle of isotropy of space, the index tensor is explicitly expressed by the elastic strain tensor and the plastic strain tensor, or by the stress tensor and the total strain tensor. The directions of polarization for a given wave-vector are parallel to the secondary principal axes of the diametral conic section of the pseudo-strain quadric, cut by a plane parallel to the wave-front. The birefringent effect is the product of the secondary principal pseudo-strain difference and a scalar function of the invariants of the elastic strain and the plastic strain. Two special deformation states are investigated. Polynomial approximations are presented. In first-order approximation, the birefringent effect is expressed by a linear combination of stress and strain, which is the formula proposed by several investigators [1, 2, 3, 4, and 6].

1. INTRODUCTION

THERE have been reported several experimental and theoretical investigations with respect to the birefringent properties of plastic solids in the elastoplastic deformation region. A formula, which indicates that the birefringent effect is expressed as a linear combination of stress and strain, has been proposed by Filon and Jessop [1], Coker and Filon [2], Bayoumi and Frankl [3], and Fujii and Tokuoka [4]. A generalization of the formula for large deformation was advanced by Tokuoka and Miyakawa [5]. But all of the above studies should be regarded as semi-empirical ones. The author presented another paper [6], in which a proposition is introduced from the microscopic point of view and the directions of polarization are specified. But the mechanical foundations of these formulae cannot be called to be too firm.

In this paper, the photo-elastoplastic medium will be defined mathematically and the general birefringent formula for such a material will be proposed.

2. DEFINITION OF PHOTO-ELASTOPLASTIC MEDIA

The birefringent properties of a non-magnetic transparent material depend completely upon the dielectric constants of the material for a particular observing wave-frequency [7 or 8].

In the system of Gaussian units the dielectric constant and the magnetic permeability of the vacuum are both unity and the magnetic permeability of all transparent materials may be assumed to be unity.

When a given artificially birefringent material is deformed, intermolecular changes of angle and distance occur and the orientation of the molecules is changed. The dielectric tensor $\boldsymbol{\varepsilon} \equiv \|\varepsilon_{ij}\|$ and the index tensor $\boldsymbol{\eta} \equiv \|\eta_{ij}\|$, which is the inverse matrix of $\boldsymbol{\varepsilon}$ and which specifies the velocity of propagation throughout the stressed material, may be considered as the macroscopic dielectric mean effects of these changes and deviations from the equilibrium positions.

Thus the birefringent properties of a given material depend completely upon the functional relationship between the index tensor and the deformation state, which may be considered as the macroscopic geometric mean results of these changes and deviations.

A deformation state of a mechanically elastoplastic material is specified by the elastic strain tensor ${}_{\mathrm{E}}\mathbf{e} \equiv \|{}_{\mathrm{E}}e_{ij}\|$ and the plastic strain tensor ${}_{\mathrm{P}}\mathbf{e} \equiv \|{}_{\mathrm{P}}e_{ij}\|$. The observable strain tensor $\mathbf{e} \equiv \|e_{ij}\|$ is assumed to be the sum of ${}_{\mathrm{E}}\mathbf{e}$ and ${}_{\mathrm{P}}\mathbf{e}$, i.e.

$$\left.\begin{aligned} \mathbf{e} &= {}_{\mathrm{E}}\mathbf{e} + {}_{\mathrm{P}}\mathbf{e}, \\ \text{or} \qquad e_{ij} &= {}_{\mathrm{E}}e_{ij} + {}_{\mathrm{P}}e_{ij}, \quad (i, j = 1, 2, 3). \end{aligned}\right\} \tag{2.1}$$

From the above mentioned considerations, we propose the following definition.

DEFINITION (*Photo-Elastoplastic Medium*) *A photo-elastoplastic medium is a continuous material such that* (1) $\boldsymbol{\eta}$ *is a continuous single-valued function of* ${}_{\mathrm{E}}\mathbf{e}$ *and* ${}_{\mathrm{P}}\mathbf{e}$, *and depends on the material coordinate* (*material inhomogeneity*), *the observing light wave-frequency, and the thermodynamical variables;* (2) $\boldsymbol{\eta}$ *does not depend explicitly on the spatial coordinate* (*spatial homogeneity*); (3) *there is no preferred direction in space* (*isotropy*); (4) *when* ${}_{\mathrm{E}}\mathbf{e} = {}_{\mathrm{P}}\mathbf{e} = 0$, $\boldsymbol{\eta}$ *reduces to* $\eta_0\mathbf{I}$, *where* η_0 *is the index coefficient in the natural undeformed state and* $\mathbf{I}$ *is the unit tensor.*

The above defined material is an ideal and hypothetical one in which the optical effect arises as a result of strain alone, and the proposed material will not necessarily provide an exact description of the complete behaviour of a real birefringent material in which time dependence will probably be a significant factor.

When a given solid is stressed in a definite manner, the material deforms along a certain deformation path and will reach a deformation state, which depends on the given material. Then, when the mechanical state uncouples from the weak electric field state, the elastic strain and the plastic strain, appearing as the arguments of the index tensor in the above definition, depend on the mechano-constitutive equation of the given material.

3. PHOTO-CONSTITUTIVE EQUATIONS

The mathematical expression of (1) and (2) of the definition is

$$\boldsymbol{\eta} = \mathbf{f}_1({}_{\mathrm{E}}\mathbf{e}, {}_{\mathrm{P}}\mathbf{e}), \tag{3.1}$$

where $\mathbf{f}_1$ is tacitly assumed to depend on the material coordinate, the observing wave-frequency, and the thermodynamical variables, and, for simplicity, they are omitted from (3.1) and the following expressions.

From (2.1) we may express (3.1) as

$$\boldsymbol{\eta} = \mathbf{f}_2({}_{\mathrm{E}}\mathbf{e}, \mathbf{e}). \tag{3.2}$$

In the mechanically pure elastic material, the mechanical constitutive equation is expressed by [9]

$$\mathbf{t} = \mathbf{g}(\mathbf{e}), \tag{3.3}$$

where $\mathbf{t} \equiv \|t_{ij}\|$ is the stress tensor and $\mathbf{g}$ is an isotropic function of $\mathbf{e}$. If our photo-elastoplastic material has the one-to-one correspondence between the state of stress and the elastic state of strain,

$$\mathbf{t} = \mathbf{h}({}_{\mathrm{E}}\mathbf{e}), \tag{3.4}$$

it can be inverted, i.e. the elastic strain tensor ${}_{\mathrm{E}}\mathbf{e}$ is a single valued function of the stress tensor $\mathbf{t}$. The Hencky body and the Prandtl–Reuss body are subjected to (3.4) in special cases, that is, for Hooke's law. Then (3.2) is reduced to

$$\boldsymbol{\eta} = \mathbf{f}_2(\mathbf{h}^{-1}(\mathbf{t}), \mathbf{e}) \equiv \mathbf{f}_3(\mathbf{t}, \mathbf{e}), \tag{3.5}$$

where $\mathbf{h}^{-1}(\mathbf{t}) = {}_{\mathrm{E}}\mathbf{e}$ is the inverse relation of (3.4).

The requirement of isotropy (3) of the definition is expressed by the condition

$$\mathbf{Q}\boldsymbol{\eta}\mathbf{Q}^{-1} = \mathbf{f}_1(\mathbf{Q}_{\mathrm{E}}\mathbf{e}\mathbf{Q}^{-1}, \mathbf{Q}_{\mathrm{P}}\mathbf{e}\mathbf{Q}^{-1}), \tag{3.6}$$

for all orthogonal transformation matrices $\mathbf{Q}(t) \equiv \|Q_{ij}(t)\|$, where $\mathbf{Q}^{-1}(t)$ is the inverse matrix of $\mathbf{Q}(t)$, i.e. $\mathbf{Q}\mathbf{Q}^{-1} = \mathbf{Q}^{-1}\mathbf{Q} = \mathbf{I}$.

According to a result of Rivlin [10], a hemitropic polynomial $\mathbf{f}_1$ of two tensor variables ${}_{\mathrm{E}}\mathbf{e}$ and ${}_{\mathrm{P}}\mathbf{e}$ admits a representation of the form

$$\begin{aligned}\mathbf{f}_1({}_{\mathrm{E}}\mathbf{e}, {}_{\mathrm{P}}\mathbf{e}) = {} & (\eta_0 + a_0)\mathbf{I} + a_{1\mathrm{E}}\mathbf{e} + a_{2\mathrm{P}}\mathbf{e} \\ & + a_{3\mathrm{E}}\mathbf{e}^2 + a_{4\mathrm{P}}\mathbf{e}^2 + a_5({}_{\mathrm{E}}\mathbf{e}_{\mathrm{P}}\mathbf{e} + {}_{\mathrm{P}}\mathbf{e}_{\mathrm{E}}\mathbf{e}) + a_6({}_{\mathrm{E}}\mathbf{e}^2{}_{\mathrm{P}}\mathbf{e} + {}_{\mathrm{P}}\mathbf{e}_{\mathrm{E}}\mathbf{e}^2) \\ & + a_7({}_{\mathrm{E}}\mathbf{e}_{\mathrm{P}}\mathbf{e}^2 + {}_{\mathrm{P}}\mathbf{e}^2{}_{\mathrm{E}}\mathbf{e}) + a_8({}_{\mathrm{E}}\mathbf{e}^2{}_{\mathrm{P}}\mathbf{e}^2 + {}_{\mathrm{P}}\mathbf{e}^2{}_{\mathrm{E}}\mathbf{e}^2),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned} a_\gamma \equiv a_\gamma(&\mathrm{tr}\,{}_{\mathrm{E}}\mathbf{e}, \mathrm{tr}\,{}_{\mathrm{E}}\mathbf{e}^2, \mathrm{tr}\,{}_{\mathrm{E}}\mathbf{e}^3, \mathrm{tr}\,{}_{\mathrm{P}}\mathbf{e}, \mathrm{tr}\,{}_{\mathrm{P}}\mathbf{e}^2, \mathrm{tr}\,{}_{\mathrm{P}}\mathbf{e}^3, \\ & \mathrm{tr}\,{}_{\mathrm{E}}\mathbf{e}_{\mathrm{P}}\mathbf{e}, \mathrm{tr}\,{}_{\mathrm{E}}\mathbf{e}^2{}_{\mathrm{P}}\mathbf{e}, \mathrm{tr}\,{}_{\mathrm{E}}\mathbf{e}_{\mathrm{P}}\mathbf{e}^2, \mathrm{tr}\,{}_{\mathrm{E}}\mathbf{e}^2{}_{\mathrm{P}}\mathbf{e}^2), \quad (\gamma = 0, 1, 2, \ldots, 8);\end{aligned} \tag{3.8}$$

when the material is in the natural undeformed state, i.e. ${}_{\mathrm{E}}\mathbf{e} = {}_{\mathrm{P}}\mathbf{e} = 0$, $a_0 = 0$ holds by (4) of the definition; and we note that, for any two matrices $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{ab} \equiv \Big\| \sum_{k=1}^{3} a_{ik}b_{kj} \Big\| \qquad \text{and} \qquad \mathrm{tr}\,\mathbf{a} \equiv \sum_{k=1}^{3} a_{kk}$$

in a Cartesian coordinate system.

If we define the pseudo-strain tensor

$$\begin{aligned}\mathbf{e}^* \equiv {} & {}_{\mathrm{E}}\mathbf{e} + \alpha_{1\mathrm{P}}\mathbf{e} + \alpha_{2\mathrm{E}}\mathbf{e}^2 + \alpha_{3\mathrm{P}}\mathbf{e}^2 + \alpha_4({}_{\mathrm{E}}\mathbf{e}_{\mathrm{P}}\mathbf{e} + {}_{\mathrm{P}}\mathbf{e}_{\mathrm{E}}\mathbf{e}) \\ & + \alpha_5({}_{\mathrm{E}}\mathbf{e}^2{}_{\mathrm{P}}\mathbf{e} + {}_{\mathrm{P}}\mathbf{e}_{\mathrm{E}}\mathbf{e}^2) + \alpha_6({}_{\mathrm{E}}\mathbf{e}_{\mathrm{P}}\mathbf{e}^2 + {}_{\mathrm{P}}\mathbf{e}^2{}_{\mathrm{E}}\mathbf{e}) \\ & + \alpha_7({}_{\mathrm{E}}\mathbf{e}^2{}_{\mathrm{P}}\mathbf{e}^2 + {}_{\mathrm{P}}\mathbf{e}^2{}_{\mathrm{E}}\mathbf{e}^2),\end{aligned} \tag{3.9}$$

where

$$\alpha_\gamma \equiv \frac{a_{\gamma+1}}{a_1} \qquad (\gamma = 1, 2, \ldots, 7); \tag{3.10}$$

the index tensor is a linear function of $\mathbf{e}^*$ such that

$$\boldsymbol{\eta} = (\eta_0 + a_0)\mathbf{I} + a_1\mathbf{e}^*. \tag{3.11}$$

The pseudo-strain tensor defined by (3.9) depends not only on a given deformation state, but also on the dielectric properties of a given medium.

By the same process, the photo-elastoplastic material subjected to (3.4), has the photo-constitutive equation:

$$\boldsymbol{\eta} = (\eta_0 + b_0)\mathbf{I} + \mathbf{s}, \tag{3.12}$$

where

$$\begin{aligned} \mathbf{s} \equiv b_1\mathbf{t} + b_2\mathbf{e} + b_3\mathbf{t}^2 + b_4\mathbf{e}^2 + b_5(\mathbf{te} + \mathbf{et}) \\ + b_6(\mathbf{t}^2\mathbf{e} + \mathbf{et}^2) + b_7(\mathbf{te}^2 + \mathbf{e}^2\mathbf{t}) \\ + b_8(\mathbf{t}^2\mathbf{e}^2 + \mathbf{e}^2\mathbf{t}^2), \end{aligned} \tag{3.13}$$

and

$$\left.\begin{aligned} b_\gamma \equiv b_\gamma(\mathrm{tr}\,\mathbf{t}, \mathrm{tr}\,\mathbf{t}^2, \mathrm{tr}\,\mathbf{t}^3, \mathrm{tr}\,\mathbf{e}, \mathrm{tr}\,\mathbf{e}^2, \mathrm{tr}\,\mathbf{e}^3, \\ \mathrm{tr}\,\mathbf{te}, \mathrm{tr}\,\mathbf{t}^2\mathbf{e}, \mathrm{tr}\,\mathbf{te}^2, \mathrm{tr}\,\mathbf{t}^2\mathbf{e}^2), \\ b_0 = 0 \quad \text{for} \quad \mathbf{t} = \mathbf{e} = 0. \end{aligned}\right\} \tag{3.14}$$

4. POLARIZATION AND BIREFRINGENCE

According to the photo-constitutive equation (3.11), the index ellipsoid

$$\boldsymbol{\xi}^\dagger \boldsymbol{\eta} \boldsymbol{\xi} = 1 \tag{4.1}$$

and the pseudo-strain quadric

$$\boldsymbol{\xi}^\dagger \mathbf{e}^* \boldsymbol{\xi} = \pm 1 \tag{4.2}$$

have the same principal directions, where the one row matrix ξ denotes a radius vector and $\boldsymbol{\xi}^\dagger$ its transposed matrix.

Consider the section of the index ellipsoid cut by the diametral plane perpendicular to the wave-vector, namely parallel to the wave-front. The section is an ellipse and has at least two principal directions which are called the secondary principal axes. These axes are parallel to the directions of polarization in this wave-front [7 or 8].

Cutting the pseudo-strain quadric by the same diametral plane produces a conic section. By relation (3.11) the principal axes of this conic section and the above mentioned ellipse are identical; and the secondary principal values of the index coefficient and the pseudo-strain are related by

$$\eta'_\alpha = \eta_0 + a_0 + a_1 e^{*\prime}_\alpha, \quad (\alpha = 1, 2), \tag{4.3}$$

where the primes indicate the secondary principal values [6].

The directions of polarization are parallel to the axes of the section of the index ellipsoid. If the standard convention of the plane of polarization is introduced which assumes that the plane of polarization contains the direction of the magnetic field and is perpendicular to the electric field, the wave-velocity polarized parallel to one secondary axis is proportional to the semi axis perpendicular to it [7, p. 18].

Then the velocities v'_α of the wave polarized along the secondary principal axis correlate with the secondary principal values of the index coefficient such that

$$v'^2_\alpha = c^2 \eta'_{\alpha+1}, \qquad (\alpha = 1, 2), \quad (\text{mod. } 2), \tag{4.4}$$

where c is the wave-velocity in a vacuum.

Then (4.3) are

$$v_\alpha'^2 = v_0^2 + c^2(a_0 + a_1 e^{*\prime}_{\alpha+1}), \qquad (\alpha = 1, 2), \quad (\text{mod. } 2), \tag{4.5}$$

where v_0 is a wave-velocity in the undeformed natural state of a given material.

The fringe-order per unit thickness is expressed by

$$N = \omega\left(\frac{1}{v_1'} - \frac{1}{v_2'}\right), \tag{4.6}$$

where ω is a particular observing wave-frequency.

From (4.5) and the polarized wave-length $\lambda_\alpha' \equiv v_\alpha'/\omega$, we have

$$N_\alpha' \equiv \omega\left(\frac{1}{v_\alpha'} - \frac{1}{v_0}\right) = \frac{1}{\lambda_\alpha'}\left(\frac{v_0 - v_\alpha'}{v_0}\right),$$

and

$$N_\alpha'\lambda_\alpha' = 1 - \sqrt{\left[1 + \left(\frac{c}{v_0}\right)^2 (a_0 + a_1 e^{*\prime}_{\alpha+1})\right]}.$$

The wave-length of visible light is about $3 \sim 8 \times 10^{-4}$ mm and the maximum value of the fringe-order for a real photo-elastoplastic material, e.g. celluloid, may be assumed to be less than ten order per mm. Then we obtain

$$\left|1 - \sqrt{\left[1 + \left(\frac{c}{v_0}\right)^2 (a_0 + a_1 e^{*\prime}_\alpha)\right]}\right| < 10^{-2},$$

which implies that

$$\left|\left(\frac{c}{v_0}\right)^2 (a_0 + a_1 e^{*\prime}_\alpha)\right| < 2 \times 10^{-2}.$$

Thus we can estimate to a good approximation that

$$\frac{1}{v_\alpha'} = \frac{1}{v_0} - \frac{c^2}{2v_0^3}(a_0 + a_1 e^{*\prime}_{\alpha+1}), \qquad (\alpha = 1, 2), \quad (\text{mod. } 2), \tag{4.7}$$

and the fringe-order per unit thickness is

$$N = A(e_1^{*\prime} - e_2^{*\prime}), \tag{4.8}$$

where

$$A \equiv \frac{c^2\omega}{2v_0^3} a_1 \equiv C a_1$$

is the photoelastic sensitivity of a given material and is a function of the invariants of ${}_E\mathbf{e}$ and ${}_P\mathbf{e}$, ω, and the thermodynamical variables.

For a material whose photo-constitutive equation is (3.12), we can deduce

$$N = C(s_1' - s_2'), \tag{4.9}$$

where s_α' are the secondary principal values of $\mathbf{s}$.

5. SPECIAL CASES

5.1 Purely elastic deformation region

In this case the plastic strain tensor vanishes identically, and (3.1) is reduced to

$$\boldsymbol{\eta} = \mathbf{f}_4(\mathbf{e}). \tag{5.1}$$

If a given material is mechanically elastic, (3.3) holds. Then the photo-constitutive equation becomes

$$\boldsymbol{\eta} = \mathbf{f}_5(\mathbf{t}). \tag{5.2}$$

Equations (5.1) and (5.2) are expressed as

$$\left.\begin{aligned} \boldsymbol{\eta} &= (\eta_0+a_0)\mathbf{I}+a_1\mathbf{e}+a_3\mathbf{e}^2 \\ \text{and}\quad \boldsymbol{\eta} &= (\eta_0+b_0)\mathbf{I}+b_1\mathbf{t}+b_3\mathbf{t}^2 \end{aligned}\right\} \tag{5.3}$$

respectively, and the birefringent effects are

$$\left.\begin{aligned} N &= Ca_1(\tilde{e}'_1-\tilde{e}'_2) \\ \text{and}\quad N &= Cb_1(\tilde{t}'_1-\tilde{t}'_2), \end{aligned}\right\} \tag{5.4}$$

where

$$\left.\begin{aligned} \tilde{\mathbf{e}} &\equiv \mathbf{e}+\frac{a_3}{a_1}\mathbf{e}^2 \\ \text{and}\quad \tilde{\mathbf{t}} &\equiv \mathbf{t}+\frac{b_3}{b_1}\mathbf{t}^2, \end{aligned}\right\} \tag{5.5}$$

are called the modified strain tensor and the modified stress tensor, respectively [11], and

$$\left.\begin{aligned} a_\gamma &\equiv a_\gamma(\operatorname{tr}\mathbf{e}, \operatorname{tr}\mathbf{e}^2, \operatorname{tr}\mathbf{e}^3), \qquad (\gamma = 0, 1, 3), \\ & a_0(0,0,0) = 0, \\ b_\gamma &\equiv b_\gamma(\operatorname{tr}\mathbf{t}, \operatorname{tr}\mathbf{t}^2, \operatorname{tr}\mathbf{t}^3), \qquad (\gamma = 0, 1, 3), \\ & b_0(0,0,0) = 0. \end{aligned}\right\} \tag{5.6}$$

5.2 The principal axes of the elastic strain quadric and the plastic strain quadric coincide, and the wave-vector is parallel to one of these axes

When the principal axes of the elastic strain quadric and the plastic strain quadric are the same, they coincide with those of the stress quadric and the total strain quadric according to (3.4).

In the usual two-dimensional photo-elastoplastic experiment, the state of stress or the state of strain are two-dimensional and the ray of light propagates perpendicular to the side-planes of a specimen, that is, the wave-front is parallel to one of the principal plane of these quadrics. Thus the secondary principal axes are identical to the principal axes of these quadrics.

Then, if the secondary principal values of ${}_{\mathrm{E}}\mathbf{e}$, ${}_{\mathrm{P}}\mathbf{e}$, $\mathbf{t}$, and $\mathbf{e}$ are equal to ${}_{\mathrm{E}}e'_\alpha$, ${}_{\mathrm{P}}e'_\alpha$, t'_α, and e'_α ($\alpha = 1, 2$), respectively, the secondary principal values of $\mathbf{e}^*$ and $\mathbf{s}$ are specified by

$$\begin{aligned} e^{*\prime}_\alpha = {}&{}_{\mathrm{E}}e'_\alpha + \alpha_{1\mathrm{P}}e'_\alpha + \alpha_{2\mathrm{E}}e'^2_\alpha + \alpha_{3\mathrm{P}}e'^2_\alpha \\ &+ 2\alpha_{4\mathrm{E}}e'_{\alpha\mathrm{P}}e'_\alpha + 2\alpha_{5\mathrm{E}}e'^2_{\alpha\mathrm{P}}e'_\alpha + 2\alpha_{6\mathrm{E}}e'_{\alpha\mathrm{P}}e'^2_\alpha \\ &+ 2\alpha_{7\mathrm{E}}e'^2_{\alpha\mathrm{P}}e'^2_\alpha, \qquad (\alpha = 1, 2) \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} s'_\alpha = {}& b_1 t'_\alpha + b_2 e'_\alpha + b_3 t'^2_\alpha + b_4 e'^2_\alpha \\ &+ 2b_5 t'_\alpha e'_\alpha + 2b_6 t'^2_\alpha e'_\alpha + 2b_7 t'_\alpha e'^2_\alpha \\ &+ 2b_8 t'^2_\alpha e'^2_\alpha, \qquad (\alpha = 1, 2) \end{aligned} \tag{5.8}$$

respectively. Substituting (5.7) into (4.8), or (5.8) into (4.9), the birefringent effects can be expressed by ${}_{\mathrm{E}}e'_\alpha$ and ${}_{\mathrm{P}}e'_\alpha$, or t'_α and e'_α.

6. POLYNOMIAL APPROXIMATIONS

We can expand the phenomenological coefficients a_γ in terms of the invariants with respect to ${}_{\mathrm{E}}\mathbf{e}$ and ${}_{\mathrm{P}}\mathbf{e}$, and b_γ in terms of the invariants with respect to $\mathbf{t}$ and $\mathbf{e}$

6.1 *Zero-th approximation*

All phenomenological coefficients are zero, thus

$$\left.\begin{aligned} \boldsymbol{\eta} &= \eta_0 \mathbf{I}, \\ &\text{and} \\ N &= 0. \end{aligned}\right\} \tag{6.1}$$

The birefringent effect does not appear.

6.2 *First-order approximation*

In this case

$$\left.\begin{aligned} &a_0 = \lambda_1 \operatorname{tr} {}_{\mathrm{E}}\mathbf{e} + \lambda_2 \operatorname{tr} {}_{\mathrm{P}}\mathbf{e}, \quad a_1 = 2\mu_{10}, \quad a_2 = 2\mu_{20}, \\ &a_3 \equiv a_4 \equiv a_5 \equiv a_6 \equiv a_7 \equiv a_8 \equiv 0; \end{aligned}\right\} \tag{6.2}$$

and

$$\left.\begin{aligned} &b_0 = \bar{\lambda}_1 \operatorname{tr} \mathbf{t} + \bar{\lambda}_2 \operatorname{tr} \mathbf{e}, \quad b_1 = 2\bar{\mu}_{10}, \quad b_2 = 2\bar{\mu}_{20}, \\ &b_3 \equiv b_4 \equiv b_5 \equiv b_6 \equiv b_7 \equiv b_8 \equiv 0. \end{aligned}\right\} \tag{6.3}$$

Then we have

$$\left.\begin{aligned} \boldsymbol{\eta} = {}&(\eta_0 + \lambda_1 \operatorname{tr} {}_{\mathrm{E}}\mathbf{e} + \lambda_2 \operatorname{tr} {}_{\mathrm{P}}\mathbf{e})\mathbf{I} \\ &+ 2\mu_{10\,\mathrm{E}}\mathbf{e} + 2\mu_{20\,\mathrm{P}}\mathbf{e}, \\ \mathbf{e}^* = {}&{}_{\mathrm{E}}\mathbf{e} + \frac{\mu_{20}}{\mu_{10}}{}_{\mathrm{P}}\mathbf{e} \end{aligned}\right\} \tag{6.4}$$

and

$$\left.\begin{aligned} \boldsymbol{\eta} &= (\eta_0+\bar{\lambda}_1 \operatorname{tr}\mathbf{t}+\bar{\lambda}_2 \operatorname{tr}\mathbf{e})\mathbf{I} \\ &\quad +2\bar{\mu}_{10}\mathbf{t}+2\bar{\mu}_{20}\mathbf{e}, \\ \mathbf{s} &= 2\bar{\mu}_{10}\mathbf{t}+2\bar{\mu}_{20}\mathbf{e}, \end{aligned}\right\} \tag{6.5}$$

where $\lambda_1, \lambda_2, \ldots, \bar{\mu}_{20}$ are material constants specified by the observing wave-frequency.

In the case specified in Section 5.2, we have

$$\left.\begin{aligned} e_\alpha^{*\prime} &= {}_{\mathrm{E}}e'_\alpha+\frac{\mu_{20}}{\mu_{10}}{}_{\mathrm{P}}e'_\alpha, \\ s'_\alpha &= 2\bar{\mu}_{10}t'_\alpha+2\bar{\mu}_{20}e'_\alpha \end{aligned}\right\} \quad (\alpha = 1,2) \tag{6.6}$$

and

$$\left.\begin{aligned} N &= A\left[({}_{\mathrm{E}}e'_1-{}_{\mathrm{E}}e'_2)+\frac{\mu_{20}}{\mu_{10}}({}_{\mathrm{P}}e'_1-{}_{\mathrm{P}}e'_2)\right], \\ N &= C_1(t'_1-t'_2)+C_2(e'_1-e'_2), \end{aligned}\right\} \tag{6.7}$$

where $C_1 \equiv 2C\bar{\mu}_{10}$ and $C_2 \equiv 2C\bar{\mu}_{20}$.

The second relation of (6.7) is the formula proposed by Filon and Jessop [1], Coker and Filon [2], Bayoumi and Frankl [3], Fujii and Tokuoka [4], and Tokuoka [6].

6.3. *Second-order approximation*

In this case,

$$\left.\begin{aligned} a_0 &= \lambda_1 \operatorname{tr}{}_{\mathrm{E}}\mathbf{e}+\lambda_2 \operatorname{tr}{}_{\mathrm{P}}\mathbf{e}+\lambda_3(\operatorname{tr}{}_{\mathrm{E}}\mathbf{e})^2+\lambda_4(\operatorname{tr}{}_{\mathrm{P}}\mathbf{e})^2 \\ &\quad +\lambda_5(\operatorname{tr}{}_{\mathrm{E}}\mathbf{e})(\operatorname{tr}{}_{\mathrm{P}}\mathbf{e})+\lambda_6 \operatorname{tr}{}_{\mathrm{E}}\mathbf{e}^2+\lambda_7 \operatorname{tr}{}_{\mathrm{P}}\mathbf{e}^2 \\ &\quad +\lambda_8 \operatorname{tr}{}_{\mathrm{E}}\mathbf{e}_{\mathrm{P}}\mathbf{e}, \\ a_1 &= 2\mu_{10}+2\mu_{11}\operatorname{tr}{}_{\mathrm{E}}\mathbf{e}+2\mu_{12}\operatorname{tr}{}_{\mathrm{P}}\mathbf{e} \\ a_2 &= 2\mu_{20}+2\mu_{21}\operatorname{tr}{}_{\mathrm{E}}\mathbf{e}+2\mu_{22}\operatorname{tr}{}_{\mathrm{P}}\mathbf{e}, \\ a_3 &= \nu_{30},\ a_4 = \nu_{31},\ a_5 = \nu_{32}, \\ a_6 &\equiv a_7 \equiv a_8 \equiv 0 \end{aligned}\right\} \tag{6.8}$$

and

$$\left.\begin{aligned} b_0 &= \bar{\lambda}_1 \operatorname{tr}\mathbf{t}+\bar{\lambda}_2 \operatorname{tr}\mathbf{e}+\bar{\lambda}_3(\operatorname{tr}\mathbf{t})^2+\bar{\lambda}_4(\operatorname{tr}\mathbf{e})^2 \\ &\quad +\bar{\lambda}_5(\operatorname{tr}\mathbf{t})(\operatorname{tr}\mathbf{e})+\bar{\lambda}_6 \operatorname{tr}\mathbf{t}^2+\bar{\lambda}_7 \operatorname{tr}\mathbf{e}^2 \\ &\quad +\bar{\lambda}_8 \operatorname{tr}\mathbf{te}, \\ b_1 &= 2\bar{\mu}_{10}+2\bar{\mu}_{11}\operatorname{tr}\mathbf{t}+2\bar{\mu}_{12}\operatorname{tr}\mathbf{e}, \\ b_2 &= 2\bar{\mu}_{20}+2\bar{\mu}_{21}\operatorname{tr}\mathbf{t}+2\bar{\mu}_{22}\operatorname{tr}\mathbf{e}, \\ b_3 &= \bar{\nu}_{30}, \qquad b_4 = \bar{\nu}_{31}, \qquad b_5 = \bar{\nu}_{32}, \\ b_6 &\equiv b_7 \equiv b_8 \equiv 0, \end{aligned}\right\} \tag{6.9}$$

where $\lambda_1, \lambda_2, \ldots, \bar{\nu}_{32}$ are material constants specified by the observing wave-frequency.

In the case specified in Section 5.2, substituting (3.10), (5.7), and (6.8) into (4.8), or substituting (5.8) and (6.9) into (4.9); we obtain the second-order approximation of the birefringent effects.

7. SUMMARY AND CONCLUSIONS

1. A hypothetical photo-elastoplastic medium is defined such that the index tensor is a function of the elastic strain tensor and the plastic strain tensor.

2. The general photo-constitutive equations are deduced by means of the principle of isotropy of space.

3. The directions of polarization for a given wave-vector are identical with the secondary principal axes of the conic section of the pseudo-strain quadric, cut by the diametral plane parallel to the wave-front.

4. The fringe-order per unit thickness is the product of the secondary principal pseudo-strain difference and a scalar function of the invariants of the elastic strain tensor and the plastic strain tensor.

5. The case in which the stress tensor is a single-valued function of the elastic strain tensor is investigated.

6. Two special deformation states, i.e. (1) purely elastic deformation, (2) the principal axes of the elastic strain quadric and the plastic strain quadric coincide and the wave-vector is parallel to one of these axes, are presented.

7. Polynomial approximations of the photo-constitutive equations and the birefringent effects are presented. When the principal axes of two quadrics coincide, the birefringent effect in first-order approximation is expressed as a linear combination of stress and strain, which coincide with the results proposed by several investigators [1, 2, 3, 4, and 6].

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Résumé—Les formules générales biréfringentes, quant à un milieu optiquement élastoplastique, sont théoriquement déduites du point de vue mécanique. Un milieu photo-élastoplastique hypothétique est défini, c'est à dire que le tenseur d'indice est une fonction du tenseur de la tension élastique et du tenseur de la tension plastique. Selon le principe de l'isotropie de l'espace, le tenseur d'indice est clairement exprimé par le tenseur de la tension élastique et par le tenseur de la tension plastique, ou bien par le tenseur de l'effort et le tenseur de la tension totale. Les directions de polarisation pour un vecteur d'ondes donné, sont parallèles aux axes secondaires principaux de la section diamètrale conique de la quadrique de la pseudo-tension, coupée par un plan parallèle à l'onde enveloppe. L'effet biréfringent est le produit de la différence de la pseudo-tension secondaire principale et d'une fonction scalaire des invariants de la tension élastique et de la tension plastique. Deux états spéciaux de déformation y sont investigués. Des approximations polynômes y sont présentées.

Dans une approximation de premier ordre, l'effet biréfringent est exprimé par une combinaison linéaire d'effort et de tension, formule proposée par plusieurs rechercheurs [1, 2, 3, 4, et 6].

Zussammenfassung—Allgemeine doppelrefringente Formeln für ein optisch elastoplastisches Medium werden theoretisch vom mechanischen Gesichtspunkt aus deduziert. Ein hypothetisches photo-elastoplastisches Medium wird definiert d.h. der Index Tensor ist eine Funktion des elastischen Belastungstensors und des plastischen Belastungstensors. Gemäss dem Prinzip der Raumisotropie wird der Index-Tensor ausdrücklich durch den elastischen Belastungs Tensor und den plastischen Belastungs Tensor, oder durch den Spannungs Tensor und den totalen Belastungstensor ausgedrückt.

Die Polarisationsrichtungen für einen gegebenen Wellenvektor laufen parallel mit den sekundären Hauptachsen des diametralen Kegelschnittes der quadratischen Pseudospannung, mittels einer Ebene parallel zur Wellenfront geschnitten. Der doppelrefringente Effekt ist das Produkt des sekundären Hauptpseudospannungs-Unterschiedes und einer skalaren Funktion der Invarianten der Elastizitätsspannung und der plastischen Spannung. Zwei besondere Fälle von Formveränderung wurden untersucht. Annäherungen polynomer Natur werden gezeigt. In der Annäherung erster Ordnung wird der doppelrefringente Effekt durch eine lineare Kombination von Spannung und Dehnung ausgedrückt; das ist die von mehreren Forschungsarbeitern vorgeschlagene Formel [1, 2, 3, 4, und 6].

Абстракт—Общие формулы двойного лучепреломления для оптически эластопластической среды теоретически разработанны с механической точки зрения. Определенна гипотетическая фото-эластопластическая среда где тензор показателя является функцией тензора эластичного напряжения и тензора пластического напряжения. Следуя принципу изотропии пространства, тензор показателя явно выражен тензорами эластичного и пластического напряжения, или тензором напряжения и тотальным тензором деформации. Направления поляризации для данного вектора волны параллельны секондарным главным осям диаметрического конического сечения псевдо-деформационной квадрики. Эффект двойного лучепреломления является произведением разницы секондарной главной псевдо-деформации и скалярной функции инвариантов эластичного напряжения и пластической деформации. Исследованны два особенных состояния деформации и представленны полиноминальные приближения. В приближении первого порядка эффект двойного лучепреломления выражен в форме линейной комбинации напряжения и деформации—формула уже предложенная другими исследователями. [1, 2, 3, 4 и 6.]